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CLOSED-FORM SOLUTIONS OF AN INFINITE BEAM UNDER IMPACT LOADING

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Abstract—Based on the Euler–Bernoulli equation for the transverse vibration of elastic beams, closed-form solutions for the transverse displacements and stresses are derived for an infinitely long beam subjected to an arbitrary (impulsive) displacement input at a given point. The method of solution employs the transform technique. The solutions are given in terms of Fresnel's integrals and elementary functions, and thus permit direct numerical evaluations. An approximate solution is also presented.

1. INTRODUCTION

In a previous paper [1] a general method was suggested for decoupling the shear and bending effects in a Timoshenko beam subjected to a concentrated force impact. It was demonstrated that a simple shear theory is adequate to predict the peak shear stresses and an associated Euler–Bernoulli beam theory can be used to predict the bending stresses for the special case of a semi-infinite beam when a step velocity is suddenly applied at its free end. This enables one to bypass the extremely tedious analytical and numerical work attendent upon the use of Timoshenko Beam Theory.

The present paper is concerned with the derivation of closed-form solutions for the associated bending problem, which is described by the Euler–Bernoulli approximation, for an infinitely long beam subjected to an arbitrary displacement (or velocity) input at a given point of the beam. More specifically the displacement input is assumed in the form of a power series in time. The corresponding shear problem renders no difficulty and is omitted from the present analysis. We point out here that the experimental results of Vigness [2] have indicated that the Euler–Bernoulli description is sufficiently accurate in treating problems of this kind.

A closed-form solution for the case of a suddenly applied step velocity has been previously derived by Bohnenblust [3], using primarily the theory developed by Boussinesq several decades ago. In most engineering applications, however, there exists always a finite duration of time during which the beam is accelerated in order to attain a certain velocity. The present formulation of assuming a suddenly applied displacement in the form of a power series in time considers this transient state and includes the Bohnenblust solution as a special case. The method of solution employs the more direct transform technique. A set of integrals, which are expressed in terms of Fresnel integrals and elementary functions, is generated for obtaining closed-form solutions.

2. FORMULATION OF BENDING PROBLEM

We consider an infinitely long beam, occupying the space $-\infty < x < \infty$, which is assumed at rest at zero time ($t = 0$). At a time greater than zero the location $x = 0$ is forced to move, and the displacement at $x = 0$ is assumed to be described by a power series in time. By the symmetry property of the beam it is only necessary to consider the part of the beam which occupies $0 \leq x < \infty$ with an additional constraint that the slope vanishes at $x = 0$.

The Euler–Bernoulli equation for the transverse displacement y of the beam with constant cross-sectional area A, moment of inertia I, mass density ρ, and Young's modulus E is:

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0 \tag{1}$$

where $a^2 = EI/\rho A$.

The initial and boundary conditions are respectively:

$$y = \frac{\partial y}{\partial t} = 0 \quad \text{when} \quad t = 0, x \geq 0 \tag{2}$$

$$y = \sum_{n=1}^{N} a_n t^n \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0 \tag{3}$$

$$\frac{\partial y}{\partial x} = 0 \qquad \text{at} \quad x = 0 \quad \text{for} \quad t > 0. \tag{4}$$

In equation (3) the coefficients a_n are known constants with $a_1 = V$ and N is an arbitrary positive integer.

Let

$$\frac{\partial^2 y}{\partial x^2} = F(t) \qquad \text{at} \quad x = 0 \quad \text{for} \quad t > 0 \tag{5}$$

where $F(t)$ is an unknown function of time and is related to the bending moment at the origin through the equation

$$M(0, t) = EIF(t).$$

Then, the Fourier transform [4, p. 114]

$$Y = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} y(x, t) \sin(\xi x)\, \mathrm{d}x \tag{6}$$

must satisfy

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + a^2 \xi^4 Y = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} a^2 \xi \left\{ \xi^2 \sum_{n=1}^{N} a_n t^n - F(t) \right\}. \tag{7}$$

By equation (2)

$$Y = \frac{\mathrm{d}Y}{\mathrm{d}t} = 0 \quad \text{when} \quad t = 0. \tag{8}$$

The solution of equation (7) satisfying equation (8) is found to be [4, p. 115]

$$Y = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} a \int_0^t \left\{ \xi \sum_{n=1}^{N} a_n \eta^n - \frac{1}{\xi} F(\eta) \right\} \sin a\xi^2 (t - \eta)\, \mathrm{d}\eta.$$

By the inversion theorem,

$$y = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty Y \sin(\xi x)\, d\xi,$$

and, on substituting Y into the above expression and integrating with respect to η, we obtain

$$\begin{aligned} y = \sum_{n=1}^{N} a_n t^n + \frac{2}{\pi}\int_0^\infty \frac{1}{\xi} \sin \xi x \{\Sigma_1(t, \xi) \\ -[\Sigma_2(t, \xi) - \Sigma_3(t, \xi)] \cos a\xi^2 t \\ -[\Sigma_4(t, \xi) - \Sigma_5(t, \xi)] \sin a\xi^2 t\}\, d\xi \\ -\frac{2a}{\pi}\int_0^\infty \int_0^t \frac{1}{\xi} \sin \xi x F(\eta) \sin a\xi^2(t-\eta)\, d\eta\, d\xi \end{aligned} \tag{9}$$

where

$$\Sigma_1(t, \xi) = \sum_{n=1}^{N/2} (-1)^n H^{(2n)} (a\xi^2)^{-2n}$$

$$\Sigma_2(t, \xi) = \sum_{n=1}^{N/2} \frac{H^{(2n)}}{(2n)!}(a\xi^2)^{-2n} \sum_{m=1}^{n} (-1)^m 2n(2n-1)\ldots(2n-2m+1)(a\xi^2 t)^{2n-2m}$$

$$\Sigma_3(t, \xi) = \sum_{n=2}^{N/2} \frac{H^{(2n-1)}}{(2n-1)!}(a\xi^2)^{-(2n-1)} \sum_{m=1}^{n-1} (-1)^m (2n-1)\ldots(2n-2m)(2\xi^2 t)^{2n-2m-1}$$

$$\Sigma_4(t, \xi) = \sum_{n=1}^{N/2} \frac{H^{(2n)}}{(2n)!}(a\xi^2)^{-2n} \sum_{m=1}^{n} (-1)^m 2n(2n-1)\ldots(2n-2m+2)(2\xi^2 t)^{2n-2m+1}$$

$$\Sigma_5(t, \xi) = \sum_{n=1}^{N/2} \frac{H^{(2n-1)}}{(2n-1)!}(a\xi^2)^{-(2n-1)} \sum_{m=1}^{n} (-1)^m (2n-1)\ldots(2n-2m+1)(a\xi^2 t)^{2n-2m},$$

with the notation

$$H = \sum_{n=1}^{N} a_n t^n,$$

$$H^{(n)} = \frac{d^n H}{dt^n}.$$

Here we have assumed, without loss of generality, that N is an even integer. (If N is odd, one simply adds one more term $a_{N+1}t^{N+1}$ to the power series with $a_{N+1} = 0$.)

The integrals in equation (9) can be shown to be uniformly convergent with respect to x for $0 \leqq x \leqq M$, M being a constant. Now, when we apply equation (4) to equation (9), the unknown function $F(t)$ is found to satisfy the Abel integral equation

$$\begin{aligned} \int_0^t \frac{F(\eta)}{\sqrt{(t-\eta)}}\, d\eta = \left(\frac{8}{\pi a}\right)^{\frac{1}{2}} \int_0^\infty \{\Sigma_1(t, \xi) - \cos a\xi^2 t[\Sigma_2(t, \xi) \\ -\Sigma_3(t, \xi)] - \sin a\xi^2 t[\Sigma_4(t, \xi) - \Sigma_5(t, \xi)]\}\, d\xi. \end{aligned} \tag{10}$$

Hence, [5]

$$F(t) = \left(\frac{8}{\pi^3 a}\right)^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \int_0^\infty \{\Sigma_1(\eta, \xi) - \cos a\xi^2\eta[\Sigma_2(\eta, \xi) - \Sigma_3(\eta, \xi)]$$

$$- \sin a\xi^2\eta[\Sigma_4(\eta, \xi) - \Sigma_5(\eta, \xi)]\} \frac{\mathrm{d}\xi\, \mathrm{d}\eta}{\sqrt{(t-\eta)}} \tag{11}$$

Knowing the function $F(t)$, equation (9) can then be integrated to give the transverse displacement y. The integration process, in general, is cumbersome, but nevertheless closed-form solutions are always obtainable for a given value of N.

3. CLOSED-FORM SOLUTION FOR $N = 2$

For $N = 2$, we find

$$\Sigma_1 = \Sigma_2 = -2a_2(a\xi^2)^{-2}$$

$$\Sigma_3 = 0$$

$$\Sigma_4 = -2a_2 t(a\xi^2)^{-1}$$

$$\Sigma_5 = -(a_1 + 2a_2 t)(a\xi^2)^{-1}.$$

Substituting the above expressions into equation (11) and integrating, we obtain

$$F(t) = -\frac{1}{a}(a_1 + 2a_2 t) \tag{12}$$

Equation (9) becomes then

$$y = \sum_{n=1}^{2} a_n t^n + \frac{2}{\pi}\int_0^\infty \sin \xi x \left\{\frac{a_1}{a\xi^3}(1 - \cos a\xi^2 t - \sin a\xi^2 t)\right.$$

$$\left. + \frac{2a_2}{a^2\xi^5}(\cos a\xi^2 t - \sin a\xi^2 t + a\xi^2 t - 1)\right\} \mathrm{d}\xi. \tag{13}$$

The integral on the right hand side of equation (13) can be simplified to give

$$y = a_1 t\{1 - I_1(z) - I_2(z)\} + a_2 t^2\{1 + I_3(z) + I_4(z)\} \tag{14}$$

where

$$I_1(z) = (1 - \pi z^2)S(z) - (1 + \pi z^2)C(z) + \pi z^2 - z\left[\cos\left(\frac{\pi}{2}z^2\right) - \sin\left(\frac{\pi}{2}z^2\right)\right]$$

$$I_2(z) = (1 + \pi z^2)S(z) + (1 - \pi z^2)C(z) + z\left[\cos\left(\frac{\pi}{2}z^2\right) + \sin\left(\frac{\pi}{2}z^2\right)\right]$$

$$I_3(z) = \frac{1}{3}\left\{(\pi^2 z^4 + 6\pi z^3 - 3)C(z) + (\pi^2 z^4 - 6\pi z^2 - 3)S(z)\right.$$

$$\left. + (\pi z^3 - 5z)\cos\left(\frac{\pi}{2}z^2\right) - (\pi z^3 + 5z)\sin\left(\frac{\pi}{2}z^2\right) - \pi^2 z^4\right\}$$

$$I_4(z) = \frac{1}{3}\Big\{(\pi^2 z^4 + 6\pi z^2 - 3)S(z) - (\pi^2 z^4 - 6\pi z^2 - 3)C(z) + (\pi z^3 + 5z)\cos\left(\frac{\pi}{2}z^2\right) + (\pi z^3 - 5z)\sin\left(\frac{\pi}{2}z^2\right) - 6\pi z^2\Big\}$$

$$z^2 = x^2/2\pi at$$

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}\eta^2\right) d\eta$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2}\eta^2\right) d\eta.$$

$C(z)$ and $S(z)$ are noted to be the Fresnel integrals whose values are tabulated in standard tables [6]. The integration process is described in the Appendix.

Using the same integration process similar closed-form solution of equation (9) for $N \geqq 4$ can also be generated.

The bending stress σ is given by

$$\sigma = Eh\frac{\partial^2 y}{\partial x^2}$$

where h denotes the distance, measured perpendicular to the neutral plane of the most remote surface point. By differentiating equation (15) twice, one finds

$$\sigma = \frac{Eh}{a}\{a_1[2C(z)-1] - 2a_2 t[1 + I_1(z) - I_2(z)]\}. \tag{15}$$

4. ALTERNATIVE FORMULATION FOR $N \geqq 4$ AND DISCUSSION

Conceptually the above formulation can be carried out for any given value of N. The analysis, however, becomes more and more involved as the value of N increases. An alternative formulation is therefore presented to approximate the solution for $N \geqq 4$, using the solution obtained in the previous section.

The solution for the case of $a_2 \neq 0$ and all other $a_n = 0$ describes the behavior of the beam when subjected to a constant acceleration input at a given point. The velocity therefore is increasing linearly with time. By a proper superposition of this solution with various values of a_2 at various times one can obtain an approximate solution for a much more general velocity input. In general, a given velocity, which is a function of time, can be approximated by finite number of straight lines as shown in Fig. 1. If we denote $y_m(x, t)$, $m = 0, 1, 2 \ldots$, to be the solution for $a_2 = \frac{1}{2}\tan\theta_m$, where θ_m is defined in Fig. 1, the solution for the polygonal velocity input is given by

$$\begin{aligned} y(x,t) &= y_0(x,t) && \text{for} \quad 0 < t \leqq t_0 \\ &= y_0(x,t) - \sum_{m=1}^{n} y_m(x, t-t_m) && \text{for} \quad t_{n-1} < t \leqq t_n. \end{aligned} \tag{16}$$

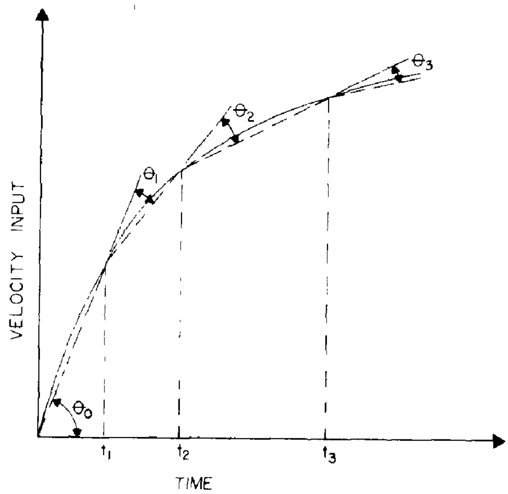

FIG. 1. Given and approximated velocity inputs.

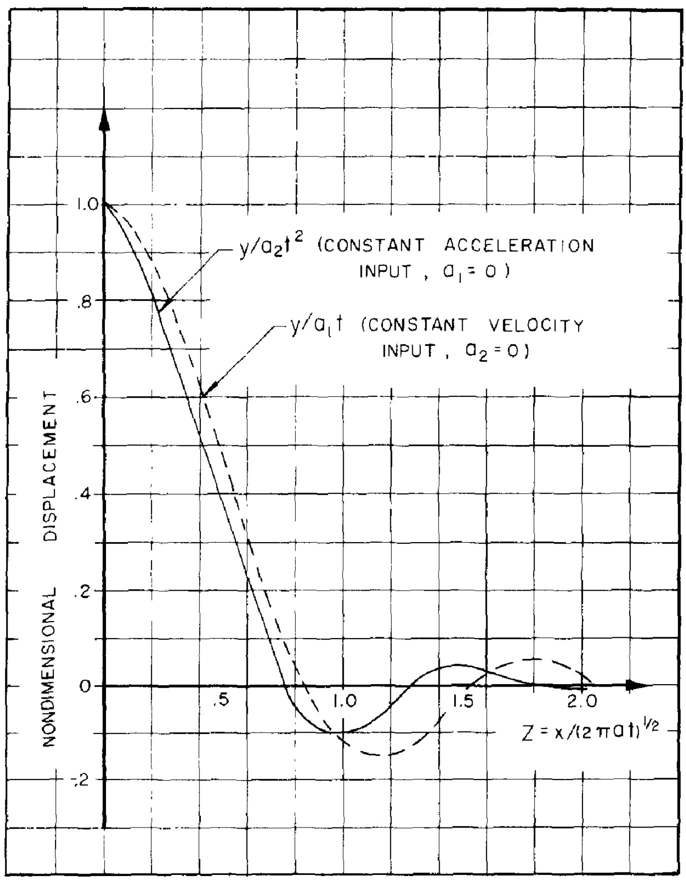

FIG. 2. Nondimensional displacements.

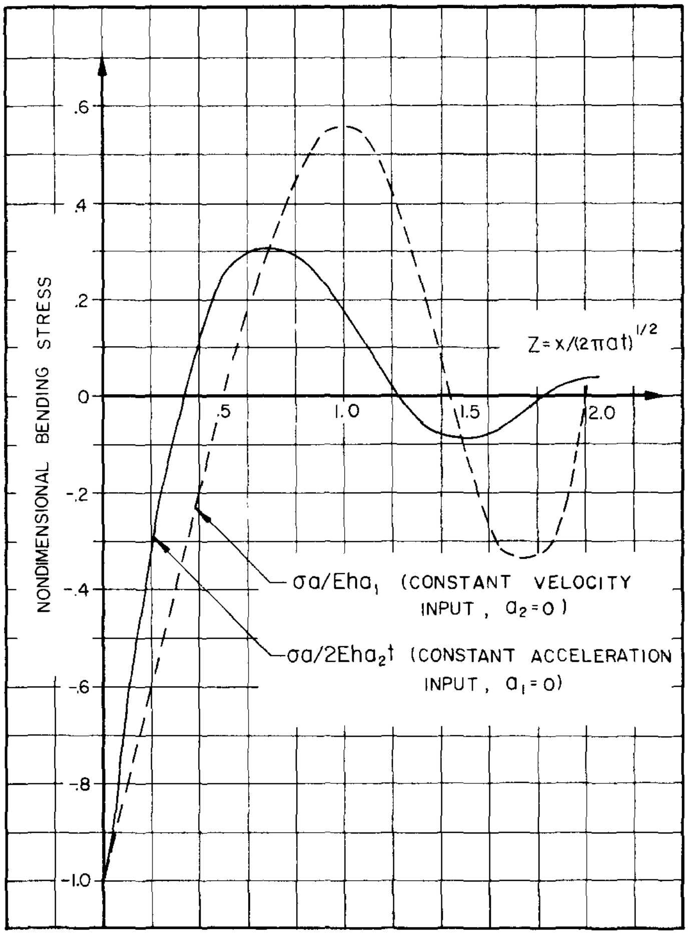

FIG. 3. Nondimensional bending stress.

As a special case of $n = 1$ and $\theta_0 = \theta_1$ one obtains the solution when a given velocity is applied to the beam at a prescribed rise time. In particular the bending stress for this case is given by

$$\sigma = -\frac{2Eha_2 t}{a}[1+I_1(z)-I_2(z)] \qquad \text{for } 0 < t \leqq t_1$$

$$= -\frac{2Eha_2}{a}\{t[1+I_1(z)-I_2(z)]-(t-t_1)[1+I_1(z_1)-I_2(z_1)]\} \quad \text{for } t > t_1 \qquad (17)$$

where $z_1 = x/\sqrt{[2\pi a(t-t_1)]}$.

For $a_1 \neq 0$ and all other $a_n = 0$ the present solution reduces to that of Bohnenblust [3] for the case of a constant velocity applied to the beam at $x = 0$. Hence, the Bohnenblust solution has been rederived from a more direct transform technique without any *a priori* assumptions on the functional form of the beam displacement.

The non-dimensional displacements and stresses for the two special cases of $a_1 \neq 0$, and $a_2 \neq 0$ are plotted respectively in Figs. 2 and 3.

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APPENDIX

We consider the integrals [4, p. 112]

$$\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos\xi^2 b\cos\xi x\,\mathrm{d}\xi = \frac{1}{4b^{\frac{1}{2}}}\left(\cos\frac{x^2}{4b}+\sin\frac{x^2}{4b}\right),$$

$$\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_0^\infty \sin\xi^2 b\cos\xi x\,\mathrm{d}\xi = \frac{1}{4b^{\frac{1}{2}}}\left(\cos\frac{x^2}{4b}-\sin\frac{x^2}{4b}\right). \tag{18}$$

If the above equations are integrated with respect to x, from 0 to x, one finds

$$\int_0^\infty \frac{\cos a\xi^2 t\sin\xi x}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}\left[C(z)+S(z)\right] \tag{19}$$

$$\int_0^\infty \frac{\sin a\xi^2 t\sin\xi x}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}\left[C(z)-S(z)\right]. \tag{20}$$

The integrations are legitimate because the improper integrals are uniformly convergent with respect to x when $0 \leqq x \leqq M < \infty$, M being a constant. Equation (19) has been previously derived in [1].

In order to integrate equation (19), we recall that

$$\int_0^\infty \frac{\sin\xi x}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2} \quad \text{for} \quad x > 0 \tag{21}$$

and write equation (19) in the form

$$\int_0^\infty \frac{\sin\xi x(\cos a\xi^2 t-1)}{\xi}\,\mathrm{d}\xi = \frac{\pi}{2}[C(z)+S(z)-1]. \tag{22}$$

Equations (20) and (22) can now be integrated twice with respect to x, from 0 to x, which results

$$\int_0^\infty \frac{\sin\xi x(\cos a\xi^2 t-1)}{\xi^3}\,\mathrm{d}\xi = \frac{\pi a t}{2}\,I_1(z) \tag{23}$$

$$\int_0^\infty \frac{\sin\xi x\sin a\xi^2 t}{\xi^3}\,\mathrm{d}\xi = \frac{\pi a t}{2}\,I_2(z). \tag{24}$$

Again, using equation (21), equation (24) can be rewritten as

$$\int_0^\infty \frac{\sin \xi x(\sin a\xi^2 t - a\xi^2 t)}{\xi^3} \mathrm{d}\xi = \frac{\pi a t}{2}[I_2(z) - 1]. \tag{25}$$

Integrating with respect to x twice, from 0 to x, we obtain

$$\int_0^\infty \frac{\sin \xi x(\cos a\xi^2 t - 1)}{\xi^5} \mathrm{d}\xi = \frac{\pi a^2 t^2}{4} I_3(z) \tag{26}$$

$$\int_0^\infty \frac{\sin \xi x(a\xi^2 t - \sin a\xi^2 t)}{\xi^5} \mathrm{d}\xi = \frac{\pi a^2 t^2}{4} I_4(z). \tag{27}$$

Equation (25) can still be integrated with respect to x, but equation (24) must be properly adjusted to assure that the result of integration exists. Continuing the process of integration, it is possible to generate integrals whose integrants involve $1/\xi^7$ or higher. These integrals will be needed in finding the closed-form solution for $N \geqq 4$.

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Résumé—Basés sur l'équation d'Euler–Bernoulli pour la vibration transversale de poutres élastiques, des solutions à formule fermée pour les déplacements et les tensions transversales sont dérivées pour une poutre infiniment longue soumise à un déplacement arbitraire (impulsif) en un point donné. La méthode de résolution employe la technique des transformations. Les solutions sont données par les intégrales de Fresnel et les fonctions élémentaires, et permettent ainsi de faire des évaluations numériques directes. Une solution approximative est aussi présentée.

Zusammenfassung—Auf Grund der Euler–Bernoulli'schen Gleichung für die Querschwingung elastischer Balken, werden Lösungen geschlossener Form abgeleitet für die Querverdrängungen und Spannungen eines unendlich langen Balkens der in einem gegebenen Punkt einer willkührlichen plötalichen Verschiebung ausgesetzt wird. Die Lösungsmethode verwendet Transformationmethoden. Die Lösung wird als Fresnel'sches Integral und Elementarfunktion gegeben, was die direkte numerische Auswertung ermöglicht. Eine Annäherungslösung wird auch gegeben.

Абстракт—На основе уравнения Эйлера-Бернулли для поперечных колебаний упругой балки, предлагается решение в замкнутом виде для поперечных перемещений и напряжений в бесконечно длинной балке, подверженной произвольным (импульсным) вводом перемещения в заданной точке. Метод решения основан на технике преобразования. Задача решается в виде интегралов френела и элементарных функций, что приводит к прóстым числовым расчетам. Даются тоже приближенное решение.